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NORMAL SUBGROUPS OF INFINITE MULTIPLY TRANSITIVE PERMUTATION GROUPS

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Some generalisations to infinite permutation groups of familiar results on normal subgroups of finite multiply transitive permutation groups are given, and the limits of these results are explored by means of examples.

1. Introduction

The purpose of this paper is to examine possible generalisations to infinite groups of well-known results on the topic of the title for finite groups. The most important such results are those of Jordan ([7] p. 65; [1] p. 198; [11] pp. 28, 31): a non-trivial normal subgroup of a k-transitive group other than S_k is $(k-1)$-transitive, except possibly if $k=3$ and the normal subgroup is a regular elementary abelian 2-group; and Burnside ([1] p. 202): a minimal normal subgroup of a 2-transitive group is either elementary abelian and regular or primitive and simple. In each case, the use of the result is somewhat limited by the scarcity of examples of finite multiply transitive groups.

Jordan's theorem extends without change to the infinite; I will prove a slightly stronger version, using Wielandt's concept of k-primitivity ([11] p. 23). The new feature, however, is that it is best possible. Using a technique of Fraïssé [3], I construct, for every k, a k-transitive group with a normal subgroup which is not k-transitive.

A different construction, due to Higman, Neumann and Neumann [5] and Cohn [2], yields a 2-transitive group with two non-abelian simple regular normal subgroups, and a 3-transitive group with two sharply 2-transitive normal subgroups. However, I show that a 4-transitive group cannot have a sharply 3-transitive normal subgroup.

Burnside's theorem clearly cannot be generalised as it stands: for minimal normal subgroups may not exist, and regular normal subgroups need not be elementary abelian. Also, the proof uses the theorem of Frobenius, that a finite transitive group with all 2-point stabilisers trivial has a regular normal subgroup (see

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[1] p. 199), and this is false for infinite groups. I show that a subgroup of finite index in an infinite 2-transitive group is primitive, while an imprimitive normal subgroup with a minimal block of imprimitivity has all 2-point stabilisers trivial.

Notation is as in Wielandt's book [11], except that G_Δ denotes the setwise stabiliser in G of the subset Δ.

2. On Jordan's theorem

A permutation group G on a set Ω is *k-primitive* if it is k-transitive and the stabiliser of any $k-1$ points acts primitively on the remaining points.

Theorem 2.1. *Let N be a non-trivial normal subgroup of the k-primitive infinite permutation group G. Then either*

(i) *N is k-transitive, or*

(ii) *$k=2$ and N is a regular elementary abelian 2-group.*

Proof. Recall that if A is a regular permutation group on Ω, then we may identify Ω with A so that A acts by right multiplication; if A is normal in G, and $1\in A$ is identified with $\alpha\in\Omega$, then the actions of $G_\alpha=G_1$ on Ω and (by conjugation) on A agree under the identification ([11] p. 27).

The theorem is proved by induction. For $k=1$, see [11] p. 17. Suppose $k=2$. Then N is transitive, and $N_\alpha\trianglelefteq G_\alpha$, so either $N_\alpha=1$ (whence N is regular) or N_α is transitive (whence N is 2-transitive). In the former case, if $x\in N$ and $x^2\neq 1$, then $\{x, x^{-1}\}$ is a block of imprimitivity for G_1; so $x^2=1$ for all $x\in N$, and N is elementary abelian.

For $k=3$, we see similarly that N is regular, sharply 2-transitive, or 3-transitive. But if N is regular, then $\{y, xy\}$ is a block of imprimitivity for G_{1x}; and if N is sharply 2-transitive, then it has a unique element x interchanging α and β, with $x^2=1$, and the orbits of x form a system of imprimitivity for $G_{\alpha\beta}$.

For $k>3$, induction shows that N is regular or k-transitive, and the first possibility is ruled out by the same argument as in the case $k=3$.

Since a 2-transitive group is primitive, we have Jordan's result:

Corollary 2.2. *Let N be a non-trivial normal subgroup of a $(k+1)$-transitive infinite permutation group G. Then the conclusions of Theorem* 2.1 *hold.*

Note that the holomorph of an elementary abelian 2-group (an affine group over GF(2)) is 3-transitive, so case (ii) occurs. The following examples show that case (i) also occurs for any value of k.

Let T be a first-order theory for which the finite models have the following two properties.

1. *Hereditary property* (HP): Any subset of a model of T (more precisely, the relational structure induced on any subset) is a model of T.

2. *Amalgamation property* (AP): If M_0, M_1, M_2 are models of T and f_i: $M_0\to M_i$ are embeddings $(i=1, 2)$, then there is a model M_3 and embeddings g_i: $M_i\to M_3$ $(i=1, 2)$ such that $f_1g_1=f_2g_2$.

Then (see Fraïssé [3], Woodrow [12]) there is, up to isomorphism, a unique countable model M of T with the properties that every finite model of T embeds in M, and every isomorphism between finite subsets of M extends to an automorphism of M. (We call such a model *homogeneous*.)

Example 2.3. Let T be the theory of k-graphs. (A *k-graph* is just a set of vertices with a collection of k-element subsets of the vertex set called edges.) Clearly T has HP and AP. Let M be the unique countable homogeneous k-graph, and $N=\text{Aut}(M)$. Then N is $(k-1)$-transitive but not k-transitive. Furthermore, N is generously $(k-1)$-transitive in the sense of Neumann [9]: the setwise stabiliser of any k-set induces the symmetric group S_k on it. Now the complementary k-graph $\overline{M}$ (whose edges are the k-sets which are not edges of M) is a countable homogeneous k-graph; so there is an isomorphism $g\colon M\to\overline{M}$. Let $G=\langle N,g\rangle$. Then N is a subgroup of G of index 2, whence normal; and G is k-transitive.

Example 2.4. Let T be the theory of k-tournaments. (A *k-tournament* is a set of vertices with a k-ary relation R having the properties that $R(x_1, \ldots, x_k)$ holds only if $x_1, \ldots, x_k$ are all distinct, and that if $x_1, \ldots, x_k$ are distinct then $R(x_1, \ldots, x_k)$ holds if and only if $R(x_2, x_1, \ldots, x_k)$ does not hold.) Again there is a unique countable homogeneous model M; this time $N=\text{Aut}(M)$ is $(k-1)$-transitive and homogeneous but not k-transitive. (The stabiliser of a k-set induces the alternating group A_k on it, so N is almost generously $(k-1)$-transitive, in Neumann's terminology.) There is an isomorphism g from M to its converse $\overline{M}$ (with $\overline{R}(x_1, \ldots, x_k)$ if and only if $x_1, \ldots, x_k$ are distinct and $R(x_1, \ldots, x_k)$ does not hold), and $G=\langle N, g\rangle$ is k-transitive and has N as a normal subgroup of index 2.

This example shows that the theorem of Ito [6] (see [9] p. 479), that for $k\geqq 4$ a normal subgroup of a k-transitive group other than S_k is generously $(k-1)$-transitive, fails for infinite groups.

3. Sharp k-transitivity

We begin this section with two further examples.

Example 3.1. Let S be a group in which all non-identity elements are conjugate. (The existence of such groups was proved by Higman, Neumann and Neumann [5].) Let $G=S\times S$ act on $\Omega=S$ by the rule $(g,h)\colon x\mapsto g^{-1}xh$. Then G is 2-transitive, and each of the two direct factors $S_1=\{x\mapsto g^{-1}x\}$ and $S_2=\{x\mapsto xh\}$ is a regular normal subgroup of G.

Example 3.2. Let D be a division ring with the property that all elements except 0 and 1 are conjugate in the multiplicative group of D. (For the existence of such division rings, see Cohn [2].) Let G be the group $\{x\mapsto a^{-1}xb+c \mid a, b, c\in D,\ ab\neq 0\}$ of permutations of $\Omega=D$. Then G is 3-transitive. Moreover, both $N_1=\{x\mapsto a^{-1}x+c \mid a, c\in D,\ a\neq 0\}$ and $N_2=\{x\mapsto xb+c \mid b, c\in D,\ b\neq 0\}$ are sharply 2-transitive normal subgroups of G.

It is clear that this sequence of examples cannot be extended two steps further, since there is no infinite sharply 4-transitive group (Tits [10]; Hall [4] p. 73). I show next that even one further step is impossible.

Theorem 3.3. *A 4-transitive infinite permutation group cannot have a sharply 3-transitive normal subgroup.*

Proof. Suppose, if possible, that G is 4-transitive on Ω, and N is a sharply 3-transitive normal subgroup of C. Let $g=(\alpha)(\beta\gamma)\ldots\in N$. Then g^2 fixes α, β and γ, so $g^2=1$. Now all involutions of N are conjugate in G (for any such involution is determined by two of its cycles, and G is 4-transitive), so all involutions fix a point. (An involution cannot fix two points, since $G_{\alpha\beta\gamma}$ centralises g and acts transitively on $\Omega-\{\alpha,\beta,\gamma\}$.) Let $g=(\alpha)(\beta\gamma)(\delta\varepsilon)\ldots$; $h=(\beta\delta)(\gamma\varepsilon)\ldots\in N$. Then g and h commute, so h fixes α. If k is another involution in G_α, say $k=(\alpha)(\beta\delta)\ldots$, then hk^{-1} fixes α, β and δ, so $h=k$. It follows that the involutions in N_α all commute and, together with the identity, form a regular normal subgroup A_α of N_α.

Now, as in Hall [4] p. 385, N_α can be identified with the group $\{x\mapsto xa+b\}$ of permutations of a nearfield F with additive group A_α and multiplicative group $N_{\alpha\beta}$. Let β and γ correspond to 0, 1 respectively of F.

Let $t=(\alpha\beta)(\gamma)\ldots\in N$. Then $\langle t\rangle=N\cap G_{\{\alpha,\beta\}\gamma}\trianglelefteq G_{\{\alpha,\beta\}\gamma}$, so $G_{\{\alpha,\beta\}\gamma}=G_{\alpha\beta\gamma}\times\langle t\rangle$. Both t and $G_{\alpha\beta\gamma}$ act on the regular group $N_{\alpha\beta}$ by conjugation; t fixes only the identity, and $G_{\alpha\beta\gamma}$ is transitive on the non-identity elements. For $x\in N_{\alpha\beta}$, $x\neq 1$, we have $x\neq x^t$, so $x^{-1}x^t\neq 1$. By transitivity of $G_{\alpha\beta\gamma}$, every element $y\neq 1$ of $N_{\alpha\beta}$ is expressible in the form $x^{-1}x^t$. We have $(x^{-1}x^t)^t=(x^{-1}x^t)^{-1}$, so t inverts $N_{\alpha\beta}$, whence $N_{\alpha\beta}$ is abelian. Now the multiplicative group of F is commutative, so F is a field. (Incidentally, putting $\alpha=\infty$, we see that t acts as $x\mapsto 1/x$, so $N\cong \mathrm{PGL}(2,F)$.)

But there is no commutative field with more than four elements whose automorphism group is transitive on elements other than 0 and 1. For suppose F were such a field. Then F has characteristic 2. The equations $t^2+t=0$, $t^2+t=1$ have between them at most four solutions, so some element $t\in F$ satisfies $t^2+t\notin\{0,1\}$. By assumption, every element $u\notin\{0,1\}$ is expressible in the form t^2+t. The map $t\mapsto t^2+t$ is GF (2)-linear and its image contains $F-\{1\}$, so its image must be F; that is, some element $t\in F$ satisfies $t^2+t=1$. But then every element of $F-\{0,1\}$ satisfies this equation, a contradiction. This completes the proof of the theorem.

4. On Burnside's theorem

In contrast to the finite case, infinite multiply transitive groups may be very richly supplied with normal subgroups. (For example, McDonough [8] found a permutation representation of the free group of rank 2 which is k-transitive for all k.) In view of this and the remarks in the Introduction, the best we could hope for is that an imprimitive normal subgroup must contain a regular normal subgroup. I know of neither a proof of, nor a counterexample to, this assertion. A couple of positive results follow.

Theorem 4.1. *A subgroup of finite index in an infinite 2-transitive group is primitive.*

Proof. It is clearly sufficient to prove the assertion for normal subgroups. So let G be 2-transitive on Ω, and N a normal subgroup of G of finite index. Suppose N is imprimitive, and let B be any block of imprimitivity. For any $\alpha,\beta\in\Omega$, there is a member of B^G containing α and β. Now G_B is transitive on B, so G is transitive on pairs (α, B'), for $\alpha\in B'\in B^G$. It follows that G_α is transitive on the members of B^G containing α. But N_α fixes one of these; so α lies in only finitely many members of B^G. It follows that B is infinite.

Let $B_1 \supseteqq B_2 \supseteqq \ldots \supseteqq B_n$ be a chain of blocks for N, and take $\alpha \in B_n$. Then N_α has fixed sets $B_n - \{\alpha\}, B_{n-1} - B_n, \ldots, B_1 - B_2, \Omega - B_1$ in $\Omega - \{\alpha\}$. But it can have at most $|G:N|$ orbits there; so $n \leqq |G:N| - 1$. Thus any descending chain of blocks is finite, so there exists a minimal block. Let B be chosen to be a minimal block. Then two points α, β lie in exactly one member of B^G, since if $\alpha, \beta \in B \cap B'$ then $B \cap B'$ is a block for N, so $B = B \cap B' = B'$ by minimality. Now, if $\gamma \notin B$, then infinitely many members of B^G contain γ and a point of B (since B is infinite), contradicting the preceding paragraph.

Theorem 4.2. *Let N be an imprimitive normal subgroup of the 2-transitive group G on Ω, having a minimal block of imprimitivity B. Then $N_{\alpha\beta} = 1$ for all $\alpha, \beta \in \Omega$ with $\alpha \neq \beta$.*

Proof. As in Theorem 4.1, any two points lie in a unique member of B^G, call members of B^G *lines*, and let $\alpha\beta$ denote the line containing α and β. Note that N_α fixes all lines containing α. For $\gamma \notin \alpha\beta$, $N_{\alpha\beta}$ fixes $\alpha\gamma$ and $\beta\gamma$, and hence fixes their intersection γ. Now $N_{\alpha\beta} \leqq N_{\alpha\gamma}$; for $\beta' \in \alpha\beta$, $\beta' \neq \alpha$, the same argument shows that $N_{\alpha\gamma}$ fixes β'. So $N_{\alpha\beta} = 1$.

Remarks. 1. Affine groups satisfy the hypotheses of the theorem, with N the subgroup consisting of translations and dilations; the lines in the proof are exactly the affine lines, and the systems of imprimitivity are the parallel classes.

2. Let G be the group of Example 3.1, and identify Ω with the regular normal subgroup S_1. Then blocks for S_1 containing the identity are exactly the subgroups of S_1; so there is no minimal block.

References

[1] W. Burnside, *Theory of Groups of Finite Order*, Dover Publ. (reprint), New York, 1955.

[2] P. M. Cohn, The embedding of firs in skew fields, *Proc. London Math. Soc.* (3), **23** (1971), 193—213.

[3] R. Fraïssé, Sur certains relations qui généralisent l'ordre des nombres rationnels, *C. R. Acad. Sci. Paris* **237** (1953), 540—542.

[4] M. Hall Jr., *The Theory of Groups*, Macmillan, New York, 1959.

[5] G. Higman, B. H. Neumann and H. Neumann, Embedding theorems for groups, *J. London Math. Soc.* (1), **26** (1949), 247—254.

[6] N. Ito, Normal subgroups of quadruply transitive groups, *Hokkaido Math. J.* **1** (1972), 1—6.

[7] C. Jordan, *Traité des Substitutions et des Equations Algébriques*, Gauthier-Villars, Paris, 1870.

[8] T. P. McDonough, A permutation representation of a free group, *Quart. J. Math. Oxford* (2), **28** (1977), 353—356.

[9] P. M. Neumann, Generosity and characters of multiply transitive permutation groups, *Proc. London Math. Soc.* (3), **31** (1975), 457—481.

[10] J. Tits, Généralisation des groupes projectifs basée sur leurs propriétés de transitivité, *Acad. Roy. Belgique Cl. Sci. Mem.* **27** (1952).

[11] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964.

[12] R. E. Woodrow, There are four countable ultrahomogeneous graphs without triangles, *J. Combinatorial Theory* (B), **27** (1979), 168—179.